4: Rotational Dynamics

We already know from **Newton's Second Law** that acceleration is directly proportional to net force and inversely proportional to mass…

$$a \propto F \quad \text{and} \quad a \propto \frac{1}{m}$$

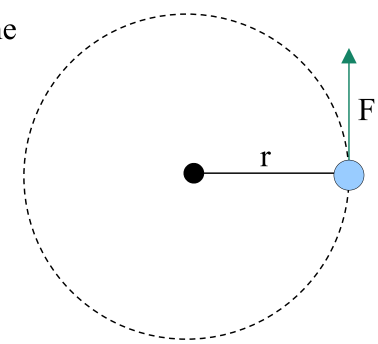
It makes sense that we would find a similar relationship between angular acceleration, torque, and mass.

- The relationship between angular acceleration and torque is pretty straight forward, looking pretty much exactly like the first part shown above...

$$\alpha \propto \tau$$

- It gets a bit more difficult to bring in the part about mass, since the ***entire object*** that is rotating has mass, with different parts of it moving differently depending on how far away from the centre of rotation.
- For this reason, we are going to have to develop a new idea for inertia (mass) spread across the entire rotating object, an idea we will call ***moment of inertia***.
- This is going to take a bit of explaining until we finally get to "moment of inertia" itself, so just follow along for the next little while.

Let's look at a simpler kind of circular motion to get an idea of how we can handle this. Imagine a mass on the end of a string being twirled around in a circle.



- Notice in the diagram we are ***NOT*** labeling the centripetal force, the force that would keep the mass moving in a circle.
- Instead, we are looking at a force that is **tangential to the circle**, and would be responsible for causing the mass to accelerate as it spins around, moving faster and faster.
  - We had a formula for this kind of acceleration a couple of lessons back, that describes tangential acceleration…

$$a = r\,\alpha$$

  - So we can re-write the classic Second Law formula a bit but substituting in this formula for tangential acceleration…

$$F = m(a)$$
$$F = m\,(r\,\alpha)$$

  - Remember that torque is really just a force acting along a lever arm distance, and the lever arm distance would be the radius in this case…

$$\tau = F\,\ell$$
$$\tau = F\,r$$

  - So take the force formula we created and **multiply both sides be radius**. The left side becomes torque, and the right side can be simplified a bit...

$$F = m (r\,\alpha)$$
$$F\,r = m (r\,\alpha)\,r$$
$$\tau = m\,r^2\,\alpha$$

- This shows us a relationship between the torque applied to the mass, and the amount of angular acceleration we would get ***if*** the mass is at ***one fixed point*** at a certain radius from the centre.
- Most importantly, the quantity **mr^2** is what we call the **moment of inertia** of the single particle spinning around. It represents all of the mass (inertia) spinning around at that distance from the centre. The moment of inertia is give the symbol I in formulas, so for a single point-mass at a distance from the centre of rotation...

$$I = mr^2$$

I = moment of inertia ($kg\,m^2$)
m = mass (kg)
r = radius (m)

Super Important!
This formula for moment of inertia can only be used for situations where the mass is a single point, or where the mass is across a thin ring at a distance equal to the radius from the centre. More on this soon!

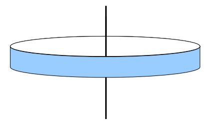
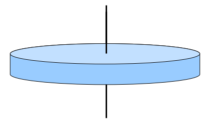
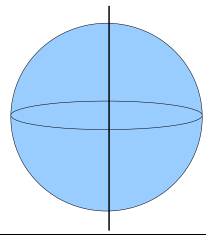
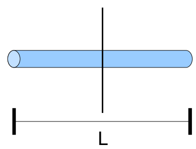
We can now create the **angular motion version of Newton's Second Law** the way it should be written.

$$\tau = I\,\alpha$$

τ = torque (Nm)
I = moment of inertia ($kg\,m^2$)
α = angular acceleration (rad/s^2)

The issue is that we are often looking at examples of something like a spinning tire, where the mass is spread out over the entire volume of the object, not just at one point at a particular radius away from the centre. This means that we can not use the moment of inertia formula shown a couple of steps above for many questions.

- In reality, we would have to consider every particle of mass at every distance from the centre of rotation in the object. Yikes! That sounds like a nasty calculus problem involving essentially an infinite number of particles.
- Good news, doing that sort of work is beyond what we would ever ask you to do in this course, so we just give the formulas for moment of inertia that have been worked out for you. They represent the standard shapes you would usually deal with, and we just assume the spread of the mass is uniform throughout the object. ***You do not have to memorize these formulas, they would be given to you if needed!***
- In all of the example diagrams, the straight line represents the axis of rotation that the object would be spinning around.
- If you use one of these formulas to calculate the moment of inertia, you can then use the angular motion formula for Newton's Second Law.

Object Type	Example	Location of Axis	**Moment of Inertia ($kg\ m^2$)**
thin hollow ring with radius, r		through centre	**$I = mr^2$**
uniform cylinder with radius, r		through centre	**$I = \frac{1}{2} mr^2$**
uniform sphere with radius, r		through centre	**$I = \frac{2}{5} mr^2$**
uniform rod with length, L		through centre	**$I = \frac{1}{12} mL^2$**
uniform rod with length, L		at one end	**$I = \frac{1}{3} mL^2$**

Example 1: A regulation bowling ball has a radius of 10.9 cm and a mass of 5.9 kg. When a bowler throws the bowling ball, they give it some rotation as they go through the throw. During a throw that takes 0.500 s, a bowler takes the ball from no rotations to 6.00 rotations per second. **Determine** the torque that was exerted on the ball.

We know it starts with an angular velocity of zero ($\omega_i = 0$). Lets start by finding what 6.00 rotations is in rads so we can have a final angular velocity of the ball.

6.00 rotations = 6.00 (2π) = 37.68 rad

$$\omega_f = \frac{\Delta \theta}{\Delta t}$$

$$\omega_f = \frac{37.68\, rad}{1\, s}$$

$$\omega_f = 37.68\, rad/s$$

The throw takes 0.500 s, so we can figure out the angular acceleration…

$$\alpha = \frac{\Delta \omega}{t} = \frac{37.68 - 0}{0.500} = 75.36\, rad/s^2$$

We need the moment of inertia for the bowling ball, a sphere...

$$I = \frac{2}{5} m r^2$$

$$I = \frac{2}{5} 5.9 (0.109)^2$$

$$I = 0.02803916 \, kg \, m^2$$

Now we can finally calculate the torque…

$$\tau = I \alpha$$

$$\tau = 0.02803916 \, (75.36)$$

$$\tau = 2.11303 = 2.1 \text{ N·m}$$